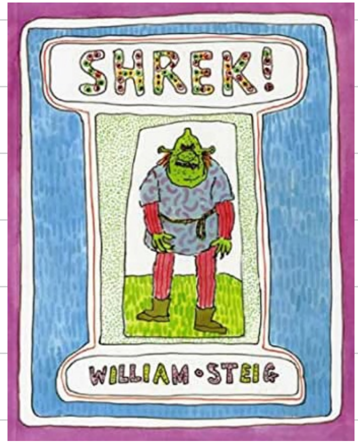
# Yiddish of the Day

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# Coordinate Vectors

- **Recall**: We say $\mathcal{B} = (v_1, \ldots, v_n)$ is a **basis** for $V$
  if
  1) $\text{span}(\mathcal{B}) = V$
  2) $\mathcal{B}$ is LI

$\Longleftrightarrow$ every vector $w \in V$ we can write $w$ ! as a LC of vectors in $\mathcal{B}$

This gives us a way to "name" vectors easilly

**Def:** Let $\mathcal{B} = (v_1 \dots v_n)$ be basis for $V$

Let $w \in V$ be arbitrary and write $w = c_1v_1 + \dots + c_nv_n$ $(c_1 \dots c_n \in \mathbb{F})$

Then we call the vector

$$[w]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{F}^n$$

the coordinate vector of $w$ with respect to basis $\mathcal{B}$

this allows us to pass questions about "abstract vectors" to coordinate vectors in $\mathbb{F}^n$

Exc: Let $\mathcal{B} = (v_1 \dots v_n)$ be basis for $V$.

try to prove this

1) A vector $w \in \operatorname{span}(z_1, \dots, z_k)$

$\iff [w]_{\mathcal{B}} \in \operatorname{span}([z_1]_{\mathcal{B}}, \dots, [z_k]_{\mathcal{B}})$

2) A list of vectors $w_1 \dots w_k$ are LI in $V$

$\iff [w_1]_{\mathcal{B}}, \dots, [w_k]_{\mathcal{B}}$ are LI in $\mathbb{F}^n$

Potential issue

- these "coordinates" are dependent on $\mathcal{B}$

- Q? What if we chose a different basis $\mathcal{B}'$

  That is, how are

$$\underbrace{\left[\quad\quad\right]_{\mathcal{B}}} \qquad \underbrace{\left[\quad\quad\right]_{\mathcal{B}'}}$$

  related?

- To answer this, we will turn to linear transformations

# Linear transformations

- What should be the "proper" def of a function between 2 $\mathbb{F}$-vs $V, W$?
  - we can add in $V, W$
  - we can scale by a $\# \in \mathbb{F}$

$\rightsquigarrow$ the functions should preserve this.

[... whispers ...... category theory]

**Def:** A function $T: V \to W$ between 2 $\mathbb{F}$-vs is a linear transformation if (linear map, linear)

1) $\forall v_1, v_2 \in V \quad T(v_1 +_V v_2) = T(v_1) +_W T(v_2)$

2) $\forall v \in V, c \in \mathbb{F} \quad T(cv) = cT(v)$

ex) i) $V = \mathbb{F}^n$, $W = \mathbb{F}^m$ and $A \in M_{m \times n}(\mathbb{F})$

Then $T(\vec{x}) := A\vec{x}$ for $\vec{x} \in \mathbb{F}^n$

$\left( A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 \text{ and } A(c\vec{x}) = cA\vec{x} \right)$

ii) Let $\begin{tikzcd} S \arrow[r, "f"] & S' \arrow[d, "g"] \\ & \mathbb{F} \end{tikzcd}$ be a function of sets.

This gives a linear map

$$\mathrm{Fct}(S', \mathbb{F}) \xrightarrow{f^*} \mathrm{Fct}(S, \mathbb{F})$$

"(\varphi)" $\quad g \longmapsto g \circ f$

(i.e $f^*(g) = g \circ f$ )

Pf) Want to check that $f^*(g_1 + g_2) = f^*(g_1) + f^*(g_2)$

Indeed $f^*(g_1 + g_2) = (g_1 + g_2) \circ f$. Now let $s \in S$.

Then $(g_1 + g_2) f(s) = g_1(f(s)) + g_2(f(s))$

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$$\underset{f^*(g_1+g_2)(s)}{\shortparallel} = (g_1 \circ f)(s) + (g_2 \circ f)(s)$$

$$= f^*(g_1)(s) + f^*(g_2)(s)$$

Thus $f^*(g_1 + g_2) = f^*(g_1) + f^*(g_2)$

Check that $f^*(cg) = c f^*(g)$ for $c \in \mathbb{F}$, $g \in \mathrm{Fct}(S', \mathbb{F})$

# Properties of linear maps

**Def**: Let $T: V \to W$ be linear. Then

1) the Kernal of $T$ is the subset

$$\ker(T) = \{v \in V \mid T(v) = 0_w\} \subseteq V$$

2) The nullity of $T$ is the dimension of $\ker(T)$

3) The range (or image) of $T$

is $\mathrm{im}(T) = \left\{ w \in W \;\middle|\; \begin{matrix} \exists v \in V \text{ with} \\ T(v) = w \end{matrix} \right\} \subseteq W$

4) The rank of $T$ is the dimension of image of $T$

Warm up exc: 1) Show $\ker(T) \subseteq V$ is a subspace

exc) 2) Show $\mathrm{im}(T) \subseteq W$ is a subspace.

Pf) Lets check that $\ker(T)$ is a subspace. Indeed, note

$$T(0_v) = T(0_v + 0_v) = T(0_v) + T(0_v)$$

$\rightsquigarrow$ add inverse $0_W = T(0_V)$ i.e. $0_V \in \ker(T)$

- Also $x_1, x_2 \in \ker(T)$ then $T(x_1 + x_2) = T(x_1) + T(x_2) = 0_W$
- Given $\alpha \in \mathbb{F}$, $x \in \ker(T)$ then $T(\alpha x) = \alpha T(x) = 0_W$ ✓

Recall: We say a function $f: X \to Y$ is injective (1-1) if $f(x) = f(y) \iff x = y$

Prop: $T: V \to W$ linear. Then $T$ is injective iff $\ker(T) = \{0_V\}$

Pf) Assume $T$ injective and let $x \in \ker(T)$. Then
$$T(x) = 0 = T(0_V) \implies x = 0_V$$

Now assume $\ker(T) = \{0_V\}$

Assume $T(x) = T(y)$ for $x, y \in V$

Note $T(x) - T(y) = 0_W$ b/c $T(x) - T(y) = T(x-y)$

$\Rightarrow T(x-y) = 0_W$ so $x - y = 0 \Rightarrow x = y$ $\square$

Recall: We say a function $f: X \to Y$ is bijective if

$\exists$ $g = f^{-1}: Y \to X$ st $f \circ g = id_Y$ $(f \circ g(y) = y \ \forall y)$ and

$g \circ f = id_X$ $(gf(x) = x \ \forall x)$

Prop: Let $T: X \to Y$ be a bijective linear map. Then

$T^{-1}$ is also linear.

Pf) See notes when uploaded

ie, $T^{-1}(y_1)=x_1$, $T^{-1}(y_2)=x_2$

Let $y_1, y_2 \in Y$. Then $\exists! x_1, x_2 \in X$ such that $T(x_1)=y_1$, $T(x_2)=y_2$. Then note

$T(x_1+x_2) = T(x_1)+T(x_2) = y_1+y_2$

That is $T^{-1}(y_1+y_2) = x_1+x_2 = T^{-1}(y_1)+T^{-1}(y_2)$

Similarly note $T(\alpha x_1) = \alpha T(x_1) = \alpha y_1$

so $T^{-1}(\alpha y_1) = \alpha x_1 = \alpha T^{-1}(x_1)$ ☺

**Def**: We call a bijective linear map $T: V \to W$ an isomorphism

and we write $V \simeq W$ (\simeq)

# Ways to think about basis through lense of linear maps

Prop: Let $V$ be fd with basis $\mathcal{B} = (v_1 \dots v_n)$
For any other vector space $W$ and vectors $y_1 \dots y_n \in W$
$\exists!$ linear map $T: V \to W$ st
$T(v_1) = y_1$, $\dots$, $T(v_n) = y_n$

(ie, we can send basis anywhere we want, and that uniquely defines the map)

Pf) Let $v \in V$ be arbitrary. Write $v = c_1 v_1 + \dots + c_n v_n$
Define $T(v) = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$

This gives a function $T: V \to W$. Need to check if linear. Let $\alpha_1, \alpha_2 \in V$

$\alpha_1 = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

$\alpha_2 = d_1 v_1 + \cdots + d_n v_n$

$$T(\alpha_1 + \alpha_2) = T((c_1+d_1)v_1 + \cdots + (c_n+d_n)v_n)$$

$$= (c_1+d_1)y_1 + \cdots + (c_n+d_n)y_n$$

$$= \underbrace{c_1 y_1 + \cdots + c_n y_n}_{} + \underbrace{d_1 y_1 + \cdots + d_n y_n}_{}$$

$$= T(\alpha_1) + T(\alpha_2)$$

Let $a \in \mathbb{F}$. then $T(a\alpha_1) = ac_1 y_1 + \cdots + ac_n y_n$

$$= a(c_1 y_1 + \cdots + c_n y_n)$$

$$= a\,T(\alpha_1)$$

Bijection of sets $\mathcal{L}(V,W) \simeq \text{Functions}(\mathcal{B}, W)$

Prop: $V$ fd with basis $\mathcal{B} = (v_1, \dots, v_n)$ and let $T: V \to W$ be a linear map. Then

1) $T$ is injective $\iff$ $(T(v_1), \dots, T(v_n))$ is LI in $W$

HW 2) $T$ surjective $\iff$ $(T(v_1), \dots, T(v_n))$ spans $W$

3) $T$ isomorphism $\iff$ $(T(v_1), \dots, T(v_n))$ is a basis

Pf) 1) Assume $T$ injective and let $0_W = c_1 T(v_1) + \dots + c_n T(v_n)$ for $c_i \in \mathbb{F}$. Then $T(c_1 v_1 + \dots + c_n v_n) = 0_W$

Since $T$ injective $0_V = c_1 v_1 + \dots + c_n v_n$

Since $(v_1, \dots, v_n)$ are basis $c_1 = c_2 = \dots = c_n = 0$.

Hence $(T(v_1), \dots, T(v_n))$ are LI

Now assume $(T(v_1), \dots, T(v_n))$ is LI and let $v \in \ker(T)$

Then $v = c_1 v_1 + \cdots + c_n v_n$. Then $T$ linear

$0_W = T(v) = T(c_1 v_1 + \cdots + c_n v_n) = c_1 T(v_1) + \cdots + c_n T(v_n)$

Yet they're LI so $c_i = 0$ so $v = 0$ $\boxed{\because}$

Cor: 1) Let $\dim(V) = n$. Then $V \cong \mathbb{F}^n$

2) 2 vectors spaces are isomorphic iff they have the same dimension

Pf) 1) Let $\mathcal{B} = (v_1, \ldots, v_n)$ be basis for $V$.

Define $\varphi_{\mathcal{B}} : \mathbb{F}^n \to V$ by $\varphi_{\mathcal{B}}(e_1) = v_1$ $\ldots$ $\varphi_{\mathcal{B}}(e_n) = v_n$

$\varphi_{\mathcal{B}}(e_2) = v_2$

2) $V, W$ same dim $(n)$

$V \cong \mathbb{F}^n \cong W$

**Rmks**: The proof of (1) tells us the following: A choice of basis amounts to choosing an isomorphism

$$\varphi_B : \mathbb{F}^n \xrightarrow{\sim} V$$

- Under this recognition

$$[v]_B = \varphi_B(v)^{-1}$$

- So, if we chose two bases then we have

$$\mathbb{F}^n \xrightarrow[\sim]{\varphi_B} V \xleftarrow[\sim]{\varphi_{B'}} \mathbb{F}^n$$

Then since $\varphi_B, \varphi_{B'}$ are bijective

$$\varphi_B([v]_B) = v = \varphi_{B'}([v]_{B'})$$

$$\begin{tikzcd} \mathbb{F}^n \arrow[r, "\varphi_B"] & V \\ & \mathbb{F}^n \arrow[u, "\varphi_{B'}"'] \arrow[ul, "\underline{\varphi_B^{-1}\varphi_{B'}}"] \end{tikzcd}$$

$$\Rightarrow [v]_B = \varphi_B^{-1}\varphi_{B'}([v]_{B'})$$

$$\Rightarrow [v]_B = \underline{P_B^{B'}}[v]_{B'}$$ for some bijective map $\underline{P_B^{B'}} : \mathbb{F}^n \to \mathbb{F}^n$

# Connection to Matrices

- Recall we started today by discussing coordinate vectors and we wanted to know how they changed if we "coordinate system" from $\mathcal{B} \to \mathcal{B}'$

We are now in a spot to answer this

Thm: Let $V \xrightarrow{T} W$ be linear and let

$\mathcal{B} = (v_1 \dots v_n)$ $\quad \mathcal{C} = (w_1 \dots w_m)$ be basis for $V, W$.

Then $\exists!$ matrix $A \in M_{m \times n}(\mathbb{F})$ such that

$$[T(v)]_{\mathcal{C}} = A\,[v]_{\mathcal{B}} \quad \forall v \in V$$

(We often write $A := [T]_{\mathcal{B}}^{\mathcal{C}}$ the matrix of $T$ wrt $\mathcal{B}, \mathcal{C}$ )

Pf) Write $T(v_1) = a_{11} w_1 + a_{12} w_2 + \cdots + a_{m1} w_m$

$T(v_2) = a_{12} w_1 + \cdots + a_{m2} w_m$

$\vdots$

$T(v_n) = a_{n1} w_1 + \cdots + a_{mn} w_m$

Define $A = [T]_{\mathcal{B}}^{\mathcal{C}} = \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{pmatrix}_{m \times n}$

Cor: Let $V$ be a vector space and let $\mathcal{B}, \mathcal{B}'$ be two basis for $V$.

Consider the map

$$(V, \mathcal{B}) \xrightarrow{id} (V, \mathcal{B}')$$

(that is $id(v) = v \quad \forall v$)

Then $[id]_{\mathcal{B}}^{\mathcal{B}'} := P_{\mathcal{B}}^{\mathcal{B}'}$ is called the

change of basis matrix and $\forall v \in V$

$$[v]_{\mathcal{B}'} = P_{\mathcal{B}}^{\mathcal{B}'} [v]_{\mathcal{B}}$$

$\rightarrow$ Justifies the name "change of basis matrix"

Cor 2: a) If $V$ is $n$-dim, $W$ $m$-dim then there is an

all linear maps from $V \to W$

isomorphism

$\mathcal{L}(V,W) \xrightarrow{\cong} M_{m \times n}(\mathbb{F})$

b) $\dim(\mathcal{L}(V,W)) = mn$

More generally: $V$, $W$ vector spaces $\mathcal{B}_V$, $\mathcal{B}_V'$ and $\mathcal{B}_W$, $\mathcal{B}_W'$ basis for $V$, $W$ respectively.

If $T: V \to W$ linear then how are

$[T]_{\mathcal{B}_V}^{\mathcal{B}_W}$ and $[T]_{\mathcal{B}_V'}^{\mathcal{B}_W'}$ related?

- Consider the composable maps

$$V \xrightarrow{T} W \xrightarrow{S} Z, \qquad \underbrace{V \to Z}_{S \circ T}$$

1) $[T]$

2) $[S]$

3) $[S \circ T] = \underline{[S][T]}$ !!!

(see HW for example)

Back to the question above: Write $(V, \mathcal{B}_v)$ to emphasize what basis using: Write $P = P_{\mathcal{B}_v}^{\mathcal{B}_v'}$ and $Q = P_{\mathcal{B}_w}^{\mathcal{B}_w'}$

$$\begin{array}{ccc} (V, \mathcal{B}_v) & \xrightarrow{T} & (W, \mathcal{B}_w) \\ \downarrow id_v & \cong & \downarrow id_w \\ (V, \mathcal{B}_v') & \xrightarrow{T} & (W, \mathcal{B}_w') \end{array}$$

$id_v(v) = v$

$id_w(w) = w$

$\longrightarrow T \circ id_v = id_w \circ T$

$\longrightarrow [T \circ id_v] = [id_w \circ T]$

$= [T]_{\mathcal{B}_v'}^{\mathcal{B}_w'} P = Q [T]_{\mathcal{B}_v}^{\mathcal{B}_w}$

$$\boxed{[T]_{B_V}^{B_W} = Q^{-1}[T]_{B_V'}^{B_W'} P}$$